(mostly the) Final Exam “Discrete Mathematics” - 2023

Firstname	
Surname	

Ex 1	Ex 2	Ex 3	Ex 4	Ex 5	Ex 6	Ex 7	Ex 8	Ex 9	Ex 10

Remarks:

- Fill in your name clearly and sign the exam
- You are not allowed to open the exam before the exam starts
- Time allocated to each student is **3 hours = 180 minutes**
- If any unauthorized electronics or preparation material is found with the student, it will lead to disqualification

Signature: ______________________

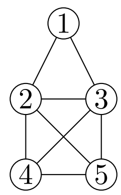
Exercise 1. Let $2 \leq d \leq n-1$. Compute the number of labeled trees with n vertices such that each vertex has degree d or 1.

Solution. This corresponds to Prufer codes of length $n-2$ in which every vertex appears 0 or $d-1$ times. If $d-1$ divides $n-2$ we have $\frac{(n-2)!}{((d-1)!)^{\frac{n-2}{d-1}}}$ such Prufer codes and $\binom{n}{\frac{n-2}{d-1}}$ chaices for he vertices of degree d. The solution is therefore

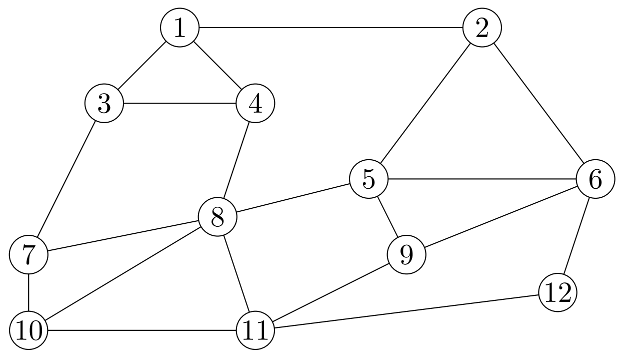
$$\text{number of labeled trees} = \begin{cases} \binom{n}{\frac{n-2}{d-1}} \frac{(n-2)!}{((d-1)!)^{\frac{n-2}{d-1}}} & \text{if } (d-1)|(n-2) \\ 0 & \text{else.} \end{cases}$$

Exercise 2. In this exercise we want to "draw graphs in one line without drawing an edge twice". This means finding a path in the graph that contains every edge exactly once. The first vertex of the path does not necessarily have to be the same as its last vertex.

(1) Assuming you want to find such a path in the following graph, at which vertices can you start? Justify your answer.



(2) Can you find such a path in the graph below? Justify your answer.



Solution. (1) We notice that there are two vertices of odd degree. These can not be in the middle (i.e. not first or last vertex) of the path because with every time the path crosses through a vertex we count +2 edges connected to this vertex. Since we have to cross all edges exactly once this count will equal the degree of the vertex. We therefore have to start/end at the two vertices with odd degree. In particular we can start at vertex 4 or 5. Indeed we can find such paths for vertex 4 and 5, namely $(4, 2, 3, 1, 2, 5, 3, 4, 5)$ and $(5, 3, 2, 1, 3, 4, 2, 5, 4)$.

(2) No. There are at least three vertices with an odd degree (for example vertex 1,2 and 3). By the above argument these can not be in the middle of the path but we can only have two vertices for start and end that can have odd degree. Contradiction.

Exercise 3. Prove the inequalities

$$\frac{2^n}{n+1} \leq \binom{n}{\lfloor n/2 \rfloor} \leq 2^{n-1}.$$

Solution. We know that the $\binom{n}{\lceil n/2 \rceil}$ is the largest of the $n+1$ binomial coefficient $\{\binom{n}{k}\}_{k=0}^{n}$. Therefore, it is at least as big as the average of all the terms. That is

$$\binom{n}{\lfloor n/2 \rfloor} \geq \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} = \frac{2^n}{n+1}. \tag{1}$$

For the other inequality, we know that

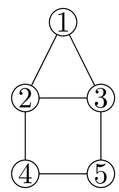
$$\sum_{k \text{ is odd}} \binom{n}{k} = \sum_{k \text{ is even}} \binom{n}{k} = \frac{2^n}{2} \tag{2}$$

Since $\binom{n}{\lceil n/2 \rceil}$ must appear in one of the two sums, the sum will be less than the RHS.

Exercise 4. Consider the graph G shown below. Let $n \in \mathbb{Z}_{\geq 0}$ and let $A(n)$ be the number of closed paths of length n in G, setting $A(0) = 5$.

(1) Show that the sequence $(A(n))_{n=0}^{\infty}$ satisfies a linear recursion.
(2) Find the formula with its initializing values for $A(n)$.

Facts from linear algebra: The trace of a matrix is the sum of its eigenvalues and the eigenvalues are the zeros of the characteristic polynomial. Note that there is no need to compute the eigenvalues.



Solution. (1) From the lecture we know that the number of closed paths of length n of a matrix is the trace of the n-th power of the adjacency matrix

$$B = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Furthermore we know that the trace is the sum of the eigenvalues of B, which are the zeros of the characteristic polynomial $x^5 = b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$ denoted by $\lambda_1, \ldots, \lambda_5$. The eigenvalues of B^n are $\lambda_1^n, \ldots, \lambda_5^n$. Then we have $A(n) = \sum_{i=1}^{5} \lambda_i^n$. Since they are the zeros of the characteristic polynomial we can use the linear recurrence theorem and obtain that

$$A(n+5) = b_4A(n+4) + b_3A(n+3) + b_2A(n+2) + b_1A(n+1) + b_0A(n)$$

and therefore satisfies a linear recurrence. Note that the linear recurrence theorem also holds if the eigenvalues have higher multiplicity since we can choose the leading coefficients of the polynomials in the explicit expression of the sequence to be 0.

(2) The characteristic polynomial is $\det(B - x\mathrm{Id}_5) = -x^5 + 6x^3 + 2x^2 - 4x$. It has 5 eigenvalues of multiplicity 1. The linear recurrence is

$$A(n+5) = 6A(n+3) + 2A(n+2) - 4A(n+1)$$

which is the same as

$$A(n+4) = 6A(n+2) + 2A(n+1) - 4A(n)$$

but starting from $n = 1$. For the starting conditions we can compute by hand (by computing the traces of B^0, B, B^2, B^3, B^4) that $A(0) = 5, A(1) = 0, A(2) = 12, A(3) = 6, A(4) = 56$.

Exercise 5. We call a permutation $(x_1, \ldots, x_{2n})$ of the numbers $1, \ldots, 2n$ pleasant if $|x_i - x_{i+1}| = n$ for at least one $i \in \{1, \ldots, 2n-1\}$. Prove that more than half of all permutations are pleasant for each positive integer n.
Hint: This is a good time to recall the inclusion-exclusion principle and the inequalities that we can derive from it.

Solution. Let A_k be the set of permutations where k and $k+n$ are in neighbouring positions. Therefore the set of all pleasant permutations is

$$A = \cup_{i=1}^{n} A_k.$$

We can count them by the inclusion exclusion principle.

$$|A| = \sum_{1\le k\le n} |A_k| - \sum_{1\le k<l\le n} |A_l \cap A_k| + \sum_{1\le k<l<m\le n} |A_l \cap A_k \cap A_m| - \ldots \tag{3}$$

$$\ge \sum_{1\le k\le n} |A_k| - \sum_{1\le k<l\le n} |A_l \cap A_k| \tag{4}$$

To count $|A_k|$ we note that the element k fixes the element $k+n$ either in the position before or after it. We therefore see them as a unit and count permutations of $2n-1$ elements. We obtain $|A_k| = 2(2n-1)!$. To count $|A_l \cap A_k|$ we group k and $k+n$ together and l and $l+n$ The case $k = l+n$ can not arise because $k \le n$. Therefore similar to the above we have $|A_l \cap A_k| = 4(2n-2)!$. Therefore

$$|A| \ge \binom{n}{1} 2(2n-1)! - \binom{n}{2} 4(2n-2)! = 2n! - (2n)(2n-2)(2n-2)! = \frac{(2n)!}{2n-2} \ge \frac{(2n)!}{2}$$

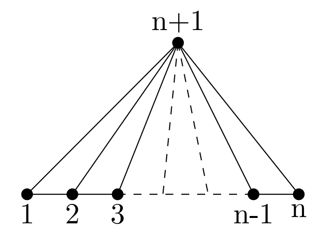
Noting that $\frac{(2n)!}{2}$ is half of all permutations we get the desired result

Remark: it also works with A_i being the set of permutations with $|x_i - x_{i+1}| = n$. In this case one has to be careful about $i = j \pm 1$ when counting the size of intersections.

Exercise 6. Let F_n be the Fibonacci numbers with $F_1 = F_2 = 1$.

(1) Show that for $n \geq 2$ we have
$$F_{2n} = 2F_{2n-2} + \sum_{i=1}^{n-2} F_{2i} + 1.$$

(2) Let $n \geq 1$. In the picture below you can see the so called fan graph for $n+1$ vertices. Show that it has F_{2n} spanning trees.



Solution. (1) We proceed by induction. For $n = 2$ we have $3 = F_4 = 2F_2 + 1 = 2 + 1$. Assume now the claim is true for n. We then have for $n+1$ that
$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n-2} + \sum_{i=1}^{n+1-2} F_{2i} + 1 = 2F_{2n} + \sum_{i=1}^{n+1-2} F_{2i} + 1$$
which proves the claim.

(2) We would like to use the above equality. we proceed by complete induction. The base case is given by $n = 1$ where we have $1 = F_2$ spanning trees. For the induction step assume that the fan graph with n vertices has $F_{2(n-1)}$ spanning trees. Then we count spanning trees for the fan graph G with $n+1$ vertices in the following way: We identify the graph with the vertices $1, \ldots, n$ and the vertex $n+1$ with the fan graph with n vertices. For this part there are therefore $F_{2(n-1)}$ spanning trees. Now a spanning tree of G can either

- include the edge $(n-1, n)$ and not $(n, n+1)$
- not include the edge $(n-1, n)$ and include $(n, n+1)$
- include the edge $(n-1, n)$ and $(n, n+1)$

In the first two cases we obtain respectively $F_2(n-1)$ spanning trees using the induction hypothesis. In the third case we know that the spanning tree does not include the edge $(n-1, n+1)$ because spanning trees do not contain cycles. We can therefore delete the vertex n and consider the edge $(n+1, n-1)$ as given. To count spanning trees for this graph we consider two cases. The spanning tree can either

- not include the edge $(n-1, n-2)$
- include the edge $(n-1, n-2)$

In the first case we have the number of spanning trees for a fan graph with $n-1$ vertices. In the second case we can reduce our graph again by one vertex, namely n. At the end of this procedure and using the equality from part (1) we obtain F_{2n} spanning trees for the graph G.

Exercise 7. Let r_n denote the number of distinct prime factors of the number n. Show that

$$\sum_{d|n} |\mu(d)| = 2^{r_n}.$$

Solution. Let $n = p_1^{a_1} p_2^{a_2} \dots p_{r_n}^{a_{r_n}}$ be the prime decomposition of n. We have that

$$|\mu(d)| = \begin{cases} 1 \text{ if } d = p_{i_1} p_{i_2} \dots p_{i_k} \text{ for } \{i_1, i_2, \dots, i_k\} \subseteq [r_n] \\ 0 \text{ if } p_i^2 \mid d \text{ for some } i \in [r_n] \end{cases} . \tag{5}$$

So we get that

$$\sum_{d|n} |\mu(d)| = \sum_{I \subseteq [r]} 1 = 2^{r_n} \tag{6}$$

To see the last equality note that we can either choose or not choose every element in $[r_n]$. Therefore we have 2 possibilities for each of the r elements, which is in total 2^{r_n}.

Exercise 8. In this exercise we consider permutations to be written as a product of disjoint cycles, for example (132)(45).

Assume that the probability is uniform on S_n. Compute the probability that a random permutation in S_n is a cycle of length n.

Solution. There are $(n-1)!$ ways to write a cycle of length n normalized to start with 1 such that all these permutations are distinct. There are $n!$ elements in S_n, therefore the probability to have a permutation that is a cycle of length n is

$$\frac{(n-1)!}{n!} = \frac{1}{n}.$$

Exercise 9. Let $R(r, s)$ be the Ramsey numbers. Show that for $r, s \geq 2$ the following inequality holds:

$$R(r, s) \leq R(r-1, s) + R(r, s-1)$$

Solution. Consider a complete graph on $R(r-1, s) + R(r, s-1)$ vertices whose edges are coloured with two colours. Pick a vertex v from the graph, and partition the remaining vertices into two sets M and N, such that for every vertex w, w is in M if edge (vw) is blue, and w is in N if (vw) is red. Because the graph has $R(r-1, s) + R(r, s-1) = |M| + |N| + 1$ vertices, it follows that either $|M| \geq R(r-1, s)$ or $|N| \geq R(r, s-1)$. In the former case, if M has a red K_s then so does the original graph and we are finished. Otherwise M has a blue K_{r-1} and so $M \cup \{v\}$ has a blue K_r by the definition of M. The latter case is analogous. Thus the claim is true and we have completed the proof.

Exercise 10. Assume that 8 friends who studied math together are on a canoe tour. They have 4 canoes (and therefore 2 people per canoe) and 7 days. Every morning they divide themselves into 4 groups of 2 people, one group for each boat.

(1) Is it possible to order them such that after the 7 days every one was with every one in a canoe?
(2) Is it always possible to find a combination like the above if they only think of this problem on day 6, and didn't pay attention to the combinations in the first 5 days (assuming they did not already share a boat twice with the same person)?

Solution. (1) Yes, the following pairing works:

day 1: (12)(34)(56)(78)
day 2: (13)(24)(57)(68)
day 3: (14)(23)(58)(67)
day 4: (15)(26)(37)(48)
day 5: (16)(25)(38)(47)
day 6: (17)(28)(35)(46)
day 7: (18)(27)(36)(45)

A more systematic way is to put 1,2,3,4 on one side and 5,6,7,8 on the other one, everything matched with the other side. Then it is 4 regular so one can find a perfect matching for the first day and then delete that matching. Then its 3 regular and we find another matching and with the same process one can find the first 4 days. Then we can write 1,2 on one side and 3,4 on the other one, connect them and have something 2 regular, same for 5,6 and 7,8 and on the last day we put 12 34 56 78.

(2) It is not. Consider

day 1: (14)(25)(36)(78)
day 2: (15)(26)(37)(84)
day 3: (16)(27)(38)(45)
day 4: (17)(28)(34)(56)
day 5: (18)(24)(35)(67)

Now 1 still has to share a boat with 2 and 3, 2 has to share it with 1 and 3, 3 hast to share it with 1 and 2. No matter which combination is choosen o day 6, the third person will not have a partner on day 6.